Polygonal Distances Theorems for Two Regular Polygons

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Abstract

There are two regular polygons $P_n(R_1)$ and $P_n(R_2)$ which correspond to the given polygonal distances – $d_1, d_2, \ldots, d_n$. It is proved that the radii R_1, R_2 of the circumcircles of the regular polygons and the n distances are connected by the system:

$$\sum_{i=1}^{n} d_i^{2m} = n\left[(R_1^2+R_2^2)^m + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}\binom{2k}{k}(R_1R_2)^{2k}((R_1^2+R_2^2)^{m-2k}\right],$$

where $m = 1, \ldots, n-1$. The solutions for the equilateral triangles and the squares are investigated. The two-parametric families of the solutions are given in both cases, when three distances are rational numbers.

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1 Introduction

The distances $d_1, d_2, \ldots, d_n$ from a point M to the vertices of a regular n-sided polygon $P_n(R)$ satisfy the following system of equations:

$$\sum_{i=1}^{n} d_i^{2m} = n\left[(R^2+L^2)^m + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}\binom{2k}{k}(RL)^{2k}(R^2+L^2)^{m-2k}\right], \quad (1.1)$$

where $m = 1, \ldots, n-1$; R is the radius of the circumcircle and L is the distance between M and the centroid of the regular polygon [2,3].

In [4,5], it is proved that there exist two regular polygons $P_n(R_1)$, $P_n(R_2)$ having the same polygon distances – $d_1, d_2, \ldots, d_n$ from the point M. Geometrical aspects are investigated, the position of M is defined and general method of constructing the second polygon is given. In the present article algebraic aspects of this problem is investigated. The systems which connects the regular polygonal distances and the sizes of both regular polygons are given. The solutions of these systems are investigated for the equilateral triangles and the squares.

2 General Theorems

The cyclic averages are defined as sums of the like even powers of the polygonal distances [2, 3]:

$$S_n^{(2)} = \frac{1}{n}\sum_{i=1}^{n} d_i^2, \dots, S_n^{(2m)} = \frac{1}{n}\sum_{i=1}^{n} d_i^{2m}.$$

The main property of the cyclic averages – they can be expressed only in terms of R and L (1.1). The first two equations of (1.1):

$$S_n^{(2)} = R^2 + L^2, \tag{2.1}$$

$$S_n^{(4)} = R^4 + L^4 + 4R^2L^2. \tag{2.2}$$

Eliminate R and L in (1.1), we get the conditions, which must be satisfied by the distance $d_1, d_2, \dots, d_n$ it they serve as the regular polygonal distances:

Theorem 2.1. *If $d_1, d_2, \dots, d_n$ satisfy*

$$S_n^{(2m)} = (S_n^{(2)})^m + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \frac{1}{2^k}\binom{m}{2k}\binom{2k}{k}\big(S_n^{(4)} - (S_n^{(2)})^2\big)^k (S_n^{(2)})^{m-2k}, \tag{2.3}$$

where $m = 3, \dots, n-1$; they are the polygonal distances for a regular n-gon.

For the equilateral triangle, there is no condition (2.3), because for P_3 exist $S_3^{(2)}$ and $S_3^{(4)}$ only. This is naturally – according to the Pompeiu theorem there is no condition for the polygonal distances – d_1, d_2, d_3, however beginning from the square such conditions exist for all regular polygons. For example, for the square holds:

$$S_n^{(6)} = S_n^{(2)}\big(3S_n^{(4)} - 2(S_n^{(2)})^2\big),$$

which turns into:

$$d_1^2 + d_3^2 = d_2^2 + d_4^2. \tag{2.4}$$

If we initially assumed, that $d_1, d_2, \dots, d_n$ are the polygonal distances for some regular polygons $P_n(R_1)$ and $P_n(R_2)$, then from (2.1) and (2.2) follow:

$$R_1 = L_2 \text{ and } R_2 = L_1;$$

and

$$S_n^{(2)} = R_1^2 + R_2^2,$$
$$S_n^{(4)} = R_1^4 + R_2^4 + 4R_1^2R_2^2;$$

solving of which gives the sizes for both polygons:

Theorem 2.2. *If $d_1, d_2, \ldots, d_n$ are the polygonal distances of the regular polygon $P_n(R_1)$, there exists the second regular polygon $P_n(R_2)$ having the same polygonal distances and the radii of the circumcircles of the polygons are solutions of the equation:*

$$2R^4 - 2R^2S_n^{(2)} + \left(S_n^{(4)} - (S_n^{(2)})^2\right) = 0.$$

Corollary 2.1. *For the regular polygons with the same polygonal distances:*

$$R_1^2 = \frac{1}{2}\left(S_n^{(2)} + \sqrt{3(S_n^{(2)})^2 - 2S_n^{(4)}}\right),$$
$$R_2^2 = \frac{1}{2}\left(S_n^{(2)} - \sqrt{3(S_n^{(2)})^2 - 2S_n^{(4)}}\right).$$

We can conclude – Theorem 2.1 finds out the relations among regular polygonal distances, while Theorem 2.2 establishes the sizes of the regular polygons. They are equivalent to

Theorem 2.3. *For two regular polygons $P_n(R_1)$, $P_n(R_2)$ with the same polygonal distances $d_1, d_2, \ldots, d_n$ is satisfied:*

$$\sum_{i=1}^{n} d_i^{2m} = n\left[(R_1^2 + R_2^2)^m + \sum_{k=1}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2k}\binom{2k}{k}(R_1R_2)^{2k}(R_1^2 + R_2^2)^{m-2k}\right],$$

where $m = 1, \ldots, n-1$.

3 Special Cases

Equilateral Triangles

If the equilateral triangles $P_3(R_1)$, $P_3(R_2)$ of the sides a_1, a_2 have the same polygonal distances – d_1, d_2, d_3, then from Theorem 2.3 follows:

$$\begin{aligned} d_1^2 + d_2^2 + d_3^2 &= a_1^2 + a_2^2, \\ d_1^4 + d_2^4 + d_3^4 &= \frac{1}{3}\,(a_1^4 + a_2^4 + 4a_1^2a_2^2). \end{aligned} \tag{$*$}$$

Denote by the symbol – $\Delta_{(d_1,d_2,d_3)}$ the area of a triangle whose sides have lengths d_1, d_2, d_3, i.e., the Pompeiu triangle. Then,

$$\begin{aligned}3(S_3^{(2)})^2 - 2S_3^{(4)} &= \frac{1}{3}\Big((d_1^2+d_2^2+d_3^2)^2 - 2(d_1^4+d_2^4+d_3^4)\Big)\\ &= \frac{16}{3}\,\Delta^2_{(d_1,d_2,d_3)}.\end{aligned}$$

For the sides:

$$a_1^2 = \frac{1}{2}\Big(d_1^2+d_2^2+d_3^2+4\sqrt{3}\,\Delta_{(d_1,d_2,d_3)}\Big), \tag{3.1}$$

$$a_2^2 = \frac{1}{2}\Big(d_1^2+d_2^2+d_3^2-4\sqrt{3}\,\Delta_{(d_1,d_2,d_3)}\Big). \tag{3.2}$$

Squares

For the squares of the sides a_1, a_2 and the same polygonal distances – d_1, d_2, d_3, d_4 from Theorem 2.3 follows:

$$\begin{aligned}d_1^2+d_2^2+d_3^2+d_4^2 &= 2(a_1^2+a_2^2),\\ d_1^4+d_2^4+d_3^4+d_4^4 &= a_1^4+a_2^4+4a_1^2a_2^2,\\ d_1^6+d_2^6+d_3^6+d_4^6 &= \frac{1}{2}\,(a_1^2+a_2^2)(a_1^4+a_2^4+8a_1^2a_2^2).\end{aligned} \tag{$**$}$$

From the Theorem 2.1:

$$\begin{aligned}8(d_1^6+d_2^6+d_3^6+d_4^6)&+(d_1^2+d_2^2+d_3^2+d_4^2)^3\\ &= 6(d_1^2+d_2^2+d_3^2+d_4^2)(d_1^4+d_2^4+d_3^4+d_4^4),\end{aligned}$$

which is equivalent to

$$3(d_1^3+d_2^3-d_3^3-d_4^3)(d_1^2+d_3^2-d_2^2-d_4^2)(d_1^2+d_4^2-d_2^2-d_3^2)=0.$$

Enumerate the vertices of the square – $A_1A_2A_3A_4$. Then only (2.4) holds.

For P_4:

$$\begin{aligned}3(S_4^{(2)})^2 - 2S_4^{(2)} &= \frac{1}{16}\Big[3(d_1^2+d_2^2+d_3^2+d_4^2)^2 - 8(d_1^4+d_2^4+d_3^4+d_4^4)\Big]\\ &= 4\Delta^2_{(d_1,\sqrt{2}\,d_2,d_3)} = 4\Delta^2_{(d_2,\sqrt{2}\,d_3,d_4)},\end{aligned}$$

so

$$d_1^2+d_3^2 = d_2^2+d_4^2 = a_1^2+a_2^2, \tag{3.3}$$

$$a_1^2 = \frac{1}{2}\,(d_1^2+d_3^2)+2\Delta_{(d_1,\sqrt{2}\,d_2,d_3)} = \frac{1}{2}\,(d_2^2+d_4^2)+2\Delta_{(d_2,\sqrt{2}\,d_3,d_4)}, \tag{3.4}$$

$$a_2^2 = \frac{1}{2}\,(d_1^2+d_3^2)-2\Delta_{(d_1,\sqrt{2}\,d_2,d_3)} = \frac{1}{2}\,(d_2^2+d_4^2)-2\Delta_{(d_2,\sqrt{2}\,d_3,d_4)}. \tag{3.5}$$

4 Solutions for Equilateral Triangles

Rewrite the system $(*)$ in equivalent form:

$$(d_1^2+d_2^2+d_3^2+a_{1,2}^2)^2 = 3(d_1^4+d_2^4+d_3^4+a_{1,2}^4), \tag{4.1}$$
$$a_2^2 = d_1^2+d_2^2+d_3^2-a_1^2. \tag{4.2}$$

The equation (4.1) is completely symmetrical, so it is possible to change:

$$d_1 \longleftrightarrow d_2 \longleftrightarrow d_3 \longleftrightarrow a_1 \quad \text{and} \quad a_1 \longleftrightarrow a_2\ ,$$

whereas the equation (4.2) gives the rule – how to construct new solutions from given one.

Theorem 4.1. *If* $(d_1, d_2, d_3, a_1, a_2)$ *is the solution of* (4.1), (4.2), *the followings are the solutions too:*

$$\Big(a_1, d_1, d_2, d_3, \sqrt{a_1^2+d_1^2+d_2^2-d_3^2}\Big), \Big(d_3, a_1, d_1, d_2, \sqrt{d_3^2+a_1^2+d_1^2-d_2^2}\Big),$$
$$\Big(d_2, d_3, a_1, d_1, \sqrt{d_2^2+d_3^2+a_1^2-d_1^2}\Big);$$

$$\Big(a_2, d_1, d_2, d_3, \sqrt{a_1^2+d_1^2+d_2^2-d_3^2}\Big), \Big(d_3, a_2, d_1, d_2, \sqrt{d_3^2+a_2^2+d_1^2-d_2^2}\Big),$$
$$\Big(d_2, d_3, a_2, d_1, \sqrt{d_2^2+d_3^2+a_2^2-d_1^2}\Big).$$

If we perform there construction for "new" solutions again we get the chain of the solutions of the (4.1), (4.2).

Let us call solution for only one triangle (for example with side – a_1) trivial, in which M is on the circumcircle of this triangle or on a line determined by a side of the triangle. If M lies on the circumcircle of the triangle of the side – a_1, by Theorem 2.1 from [5], follows:

$$a_1 = a_2.$$

By Van Schooten theorem $d_3 = d_1 + d_2$, and rational parametrization of the equation:

$$d_1^2+d_2^2+d_1d_2 = a_1^2,$$

gives the solution of (4.1), (4.2):

$$\big(t^2-1, 1+2t, t^2+2t, t^2+t+1, t^2+t+1\big). \tag{4.3}$$

where $t \in \mathbb{Q}$.

If M lies on the line determined by the side a_1 of the first triangle, for the second triangle with side a_2, this case is not trivial and the solutions of

(4.1), (4.2) are:

$$\begin{aligned}
&\left(t^2+t+1, t^2-1, 1+2t, t^2+2t, \sqrt{t^4-2t^3+t^2+6t+3}\,\right),\\
&\left(t^2+2t, t^2+t+1, t^2-1, 1+2t, \sqrt{3t^4+6t^3+t^2-2t+1}\,\right),\\
&\left(1+2t, t^2+2t, t^2+t+1, t^2-1, \sqrt{t^4+6t^3+13t^2+6t+1}\,\right).
\end{aligned} \tag{4.4}$$

For example, for $t=2$, from (4.3) solution is

$$(3, 5, 8, 7, 7),$$

and Theorem 4.1 gives new solutions:

$$\left(7, 3, 5, 8, \sqrt{19}\,\right), \quad \left(8, 7, 3, 5, \sqrt{97}\,\right), \quad \left(5, 8, 7, 3, \sqrt{129}\,\right).$$

By using the Theorem 4.1 again, for "new" solutions, we get:

$$\begin{aligned}
&\left(\sqrt{19}\,, 7, 3, 5, \sqrt{52}\,\right), \quad \left(5, \sqrt{19}\,, 7, 3, \sqrt{84}\,\right), \quad \left(3, 5, \sqrt{19}\,, 7, 2\right);\\
&\left(\sqrt{97}\,, 8, 7, 3, \sqrt{201}\,\right), \quad \left(3, \sqrt{97}\,, 8, 7, 11\right), \quad \left(7, 3, \sqrt{97}\,, 8, \sqrt{91}\,\right);\\
&\left(\sqrt{129}\,, 5, 8, 7, 13\right), \quad \left(7, \sqrt{129}\,, 5, 8, \sqrt{139}\,\right), \quad \left(8, 7, \sqrt{129}\,, 5, \sqrt{217}\,\right).
\end{aligned}$$

Parametrization

In general case, if

$$\Delta_{(d_1,d_2,d_3)} \neq 0$$

and

$$d_1, d_2, d_3, a_1^2, a_2^2 \in \mathbb{Q},$$

from (3.1) and (3.2) follow – the area of the Pompeiu triangle must be rational multiple of $\sqrt{3}$, so the height of the side (for example d_2) is rational multiple of $\sqrt{3}$ too. To devide each side by this rational multiple we get rational triangle which is similar to origin one (see Fig. 1).

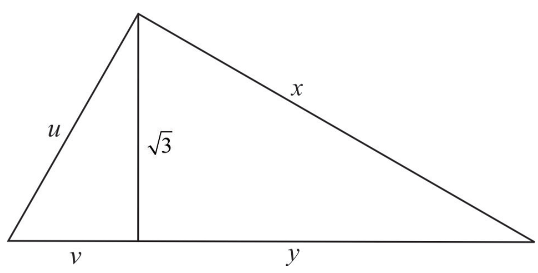

Figure 1.

The parts v and y are rationals too. The rational parametrization of the equation

$$x^2 - 3 = y^2$$

is given by formulae:

$$x = \frac{2(\xi^2 - \xi + 1)}{\xi^2 - 1}, \quad y = \frac{\xi^2 - 4\xi + 1}{\xi^2 - 1};$$

so complete rational parametrization of the rational-sided triangles with area rational multiple of $\sqrt{3}$ is given:

$$\frac{2(\xi^2 - \xi + 1)}{\xi^2 - 1} \cdot h, \quad \frac{2(\zeta^2 - \zeta + 1)}{\zeta^2 - 1} \cdot h,$$
$$\Big(\frac{\xi^2 - 4\xi + 1}{\xi^2 - 1} + \frac{\zeta^2 - 4\zeta + 1}{\zeta^2 - 1}\Big) \cdot h,$$

where $\xi, \zeta, h \in \mathbb{Q}$. The area equals:

$$\frac{\sqrt{3}}{2}\Big(\frac{\xi^2 - 4\xi + 1}{\xi^2 - 1} + \frac{\zeta^2 - 4\zeta + 1}{\zeta^2 - 1}\Big) \cdot h^2.$$

After removing the denominators and the common factor, we obtain:

Theorem 4.2. *Complete parametrization of the system* $(*)$*, when*

$$d_1, d_2, d_3, a_1^2, a_2^2 \in \mathbb{Q}$$

is given by formulae:

$$\begin{aligned} d_1 &= 2(\xi^2 - \xi + 1)(\zeta^2 - 1),\\ d_3 &= 2(\zeta^2 - \zeta + 1)(\xi^2 - 1),\\ d_2 &= (\xi^2 - 4\xi + 1)(\zeta^2 - 1) + (\zeta^2 - 4\zeta + 1)(\xi^2 - 1); \end{aligned}$$

$$\Delta_{(d_1,d_2,d_3)} = \frac{\sqrt{3}}{2}\,(\xi^2 - 1)(\zeta^2 - 1)\,d_2,$$

$$a_{1,2}^2 = \frac{1}{2}\Big(d_1^2 + d_2^2 + d_3^2 \pm 4\sqrt{3}\,\Delta_{(d_1,d_2,d_3)}\Big).$$

For

$$\zeta = -\xi,$$

is obtained the following one-parametric family of the solution of system $(*)$.

Corollary 4.1. *The solution of the system* $(*)$ *is:*

$$\begin{aligned} d_1 &= \xi^2 - \xi + 1,\\ d_3 &= \xi^2 + \xi + 1,\\ d_2 &= \xi^2 + 1,\\ a_1^2 &= \xi^2(3\xi^2 + 4),\\ a_2^2 &= 4\xi^2 + 3. \end{aligned}$$

For rationality of a_2 substitute

$$\xi = \frac{1}{2}\,\frac{t^2-4t+1}{t^2-1}\,,$$

then

$$a_2 = \frac{2(t^2-t+1)}{t^2-1}\,,$$

and we get well-known A. Kemnitz parametrization [1].

The condition of rationality for a_1:

$$\xi = \frac{4t}{3t^2-1}\,,$$

after changing $t \to \frac{1}{t}$, gives new solutions of $(*)$:

$$\begin{aligned}
d_1 &= t^4+4t^3+10t^2-12t+9,\\
d_2 &= t^4-4t^3+10t^2+12t+9,\\
d_3 &= t^4+10t^2+9,\\
a_1 &= 8t(t^2+3),\\
a_2^2 &= 3t^4+46t^2+27.
\end{aligned}$$

The smallest nontrivial integer solution of (4.1) is given, for $t=2$:

$$d_1 = 73, \;\; d_2 = 57, \;\; d_3 = 65, \;\; a_1 = 112.$$

The corresponding solutions for the system $(*)$ are:

$$\begin{aligned}
&\left(73, 57, 65, 112, \sqrt{259}\,\right),\\
&\left(112, 73, 57, 65, \sqrt{16897}\,\right),\\
&\left(65, 112, 73, 57, \sqrt{18849}\,\right),\\
&\left(57, 65, 112, 73, \sqrt{14689}\,\right).
\end{aligned}$$

5 Solutions for Squares

The system $(**)$ is equivalent to:

$$(d_1^2+d_2^2+d_3^2+d_4^2+2a_{1,2}^2)^2 = 4(d_1^4+d_2^4+d_3^4+d_4^4+3a_{1,2}^4), \tag{5.1}$$

$$d_1^2+d_3^2 = d_2^2+d_4^2, \tag{5.2}$$

$$a_2^2 = d_1^2+d_3^2-a_1^2. \tag{5.3}$$

Equations (5.1), (5.2) are equivalent to the four-distance problem. The four-distance problem is a long-open problem which asks whether there is a point in the plane at rational polygonal distances from the vertices of the square with rational side. It is believed that no such point exists. Equation (5.1) is equivalent to the three-distance problem: finding rational distances d_1, d_2, d_3 to the vertices of the square with rational side. For some time, it was belived there does not exist a solution to the three-distance problem that is not on a side of the square (trivial case). However, a one-parameter family of solutions was found by J. H. Hunter. The three and four distance problems are reviewed in § D19 of [1].

We seek the solutions of the equations (5.1)–(5.3) when three distances – d_1, d_2, d_3 are rationals and d_4^2, a_1^2, a_2^2 are rationals too.

From the symmetry, it is possible to change:

$$d_1 \longleftrightarrow d_3\ , \quad d_2 \longleftrightarrow d_4 \quad \text{and} \quad a_1 \longleftrightarrow a_2\ .$$

From the equations (5.1)–(5.3):

Theorem 5.1. *If* $(d_1, d_2, d_3, d_4, a_1, a_2)$ *is the solution of the system* (5.1)–(5.3)*, the followings are the solutions too:*

$$(d_1, a_1, d_3, a_2, d_2, d_4) \ \ \textit{and} \ \ (a_1, d_2, a_2, d_4, d_1, d_3).$$

Unfortunately for the squares, it is impossible to construct the chain of the solutions.

In trivial case, when the point M lies on the side of one square by rational parametrization of the Pythagorean equation is obtained:

$$d_1 = 1 + t^2, \ \ d_2 = 1 - t^2, \ \ d_3 = t^2 + 2t - 1, \ \ a_1 = 2t;$$
$$d_4^2 = t^4 + 4t^3 + 6t^2 - 4t + 1, \ \ a_2^2 = 2(t^4 + 2t^3 - 2t + 1).$$

In general case from (3.4) and (3.5) follows – the area of the triangle with length of sides – d_1, $\sqrt{2}\,d_2$, d_3 must be rational number, so the height of the side – $\sqrt{2}\,d_2$ is rational multiple of $\sqrt{2}$. To divide each side by this rational multiple we get rational triangle which is similar to origin one (see Fig. 2).

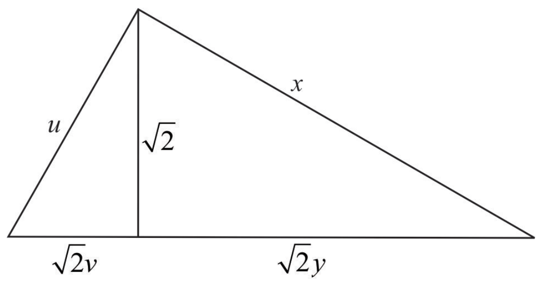

Figure 2.

The values of v and y are rationals too. The rational parametrization of the equation:

$$x^2 - 2 = 2y^2,$$

is given by formulae:

$$x = \frac{2\xi^2 - 4\xi + 4}{\xi^2 - 2}\,, \quad y = \frac{\xi^2 - 4\xi + 2}{\xi^2 - 2}\,,$$

so complete parametrization of the triangle with sides d_1, $\sqrt{2}\,d_2$, d_3 and rational area is given:

$$\frac{2\xi^2 - 4\xi + 4}{\xi^2 - 2} \cdot h, \quad \frac{2\zeta^2 - 4\zeta + 4}{\zeta^2 - 2} \cdot h,$$
$$\sqrt{2}\left(\frac{\xi^2 - 4\xi + 2}{\xi^2 - 2} + \frac{\zeta^2 - 4\zeta + 2}{\zeta^2 - 2}\right) \cdot h,$$

where $\xi, \zeta, h \in \mathbb{Q}$. The area equals:

$$\left(\frac{\xi^2 - 4\xi + 2}{\xi^2 - 2} + \frac{\zeta^2 - 4\zeta + 2}{\zeta^2 - 2}\right) \cdot h^2.$$

We obtain:

Theorem 5.2. *Complete parametrization of the system* $(**)$, *when*

$$d_1, d_2, d_3, d_4^2, a_1^2, a_2^2 \in \mathbb{Q}$$

is given by formulae

$$\begin{aligned}
d_1 &= 2(\xi^2 - 2\xi + 2)(\zeta^2 - 2),\\
d_3 &= 2(\zeta^2 - 2\zeta + 2)(\xi^2 - 2),\\
d_2 &= (\xi^2 - 4\xi + 2)(\zeta^2 - 2) + (\zeta^2 - 4\zeta + 2)(\xi^2 - 2),\\
d_4^2 &= d_1^2 + d_3^2 - d_2^2;
\end{aligned}$$

$$\Delta_{(d_1, \sqrt{2}\,d_2, d_3)} = (\xi^2 - 2)(\zeta^2 - 2)d_2,$$

$$a_{1,2}^2 = \frac{1}{2}\,(d_1^2 + d_3^2) \pm 2\Delta_{(d_1, \sqrt{2}\,d_2, d_3)}.$$

Again, for

$$\zeta = -\xi,$$

we get the one-parametric family of the solution.

Corollary 5.1. *The solution of the system* $(**)$ *is:*

$$\begin{aligned}
d_1 &= \xi^2 - 2\xi + 2,\\
d_3 &= \xi^2 + 2\xi + 2,\\
d_2 &= \xi^2 + 2,\\
d_4^2 &= \xi^4 + 12\xi^2 + 4,\\
a_1^2 &= 2\xi^2(\xi^2 + 4),\\
a_2^2 &= 8(\xi^2 + 1).
\end{aligned}$$

For rationality of a_1 substitute:

$$\xi = \frac{2(2t^2 - 4t + 1)}{2t^2 - 1},$$

then

$$a_1 = \frac{8(2t^2 - 4t + 1)(2t^2 - 2t + 1)}{(2t^2 - 1)^2}.$$

Changing $t \to \frac{1}{1-t}$, we get well-known J. H. Hunter solution [1].

The condition of rationality of a_2 is:

$$\xi = \frac{2t^2 - 4t + 1}{2t^2 - 1},$$
$$a_2 = \frac{4(2t^2 - 2t + 1)}{2t^2 - 1}.$$

Changing $t \to \frac{1}{t}$, gives new parametrization for three-distance problem:

$$d_1 = 5t^4 - 16t^3 + 12t^2 + 4,$$
$$d_3 = t^4 + 12t^2 - 32t + 20,$$
$$d_2 = 3t^4 - 8t^3 + 12t^2 - 16t + 12,$$
$$a_2 = 4(t^2 - 2t + 2)(t^2 - 2).$$

For $t = \frac{1}{2}$, there is a point at distances 85, 99 and 113 from three consecutive vertices of a square of side 140. This solution is given in [1].

For $t = \frac{3}{2}$, we get the smaller nontrivial integer solution of three-distance problem:

$$d_1 = 37, \quad d_2 = 51, \quad d_3 = 65, \quad a_2 = 20.$$

The corresponding solutions of $(**)$ are:

$$\left(37, 51, 65, \sqrt{2993}, 20, \sqrt{5194}\right), \quad \left(37, 20, 65, \sqrt{5194}, 51, \sqrt{2993}\right),$$
$$\left(20, 51, \sqrt{5194}, \sqrt{2993}, 37, 65\right).$$

References

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